Ejercicios Electrodinámica Cuántica. Capítulo 12

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8 de agosto de 2023

1. Demostrar $\varepsilon_r^* \cdot \varepsilon_s = g_{rs}$ para la base de helicidad.

La base de helicidad viene dada por la fórmula 12.2 del formulario

$$\varepsilon_0 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \ \varepsilon_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos\phi - i\sin\phi \\ \cos\theta\sin\phi + i\cos\phi \\ -\sin\theta \end{pmatrix}, \ \varepsilon_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix}, \ \varepsilon_3 = \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix}$$

Empezamos por calcular $\varepsilon_0^* \cdot \varepsilon_0$;

$$\varepsilon_0^* \cdot \varepsilon_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = 1 = g_{00}$$

Por otra parte, tenemos que

$$\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ a \\ b \\ c \end{pmatrix} = 0$$

Por lo que hemos comprobado los productos $\varepsilon_0^* \cdot \varepsilon_r = g_{0r}$. Debido a que los productos son simétricos, solo tenemos que calcular los productos con $s \geq r$;

$$\begin{aligned}
\varepsilon_1^* \cdot \varepsilon_1 &= \frac{1}{2} \begin{pmatrix} 0 & \cos\theta\cos\phi + i\sin\phi & \cos\theta\sin\phi - i\cos\phi & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta\cos\phi - i\sin\phi \\ \cos\theta\sin\phi + i\cos\phi \\ -\sin\theta \end{pmatrix} \\
&= -\frac{1}{2} \left([\cos^2(\theta)\cos^2(\phi) + \sin^2(\phi)] + [\cos^2(\theta)\sin^2(\phi) + \cos^2(\phi)] + \sin^2(\theta) \right) \\
&= -\frac{(\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)) + (\sin^2(\phi) + \cos^2(\phi))}{2} = -1
\end{aligned}$$

$$\begin{aligned}
\varepsilon_1^* \cdot \varepsilon_2 &= \frac{1}{2} \begin{pmatrix} 0 & \cos\theta\cos\phi + i\sin\phi & \cos\theta\sin\phi - i\cos\phi & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix} \\
&= -\frac{1}{2} \big([\cos^2(\theta)\cos^2(\phi) - \sin^2(\phi) + 2i\cos(\theta)\cos(\phi)\sin(\phi)] \\
&\quad + [\cos^2(\theta)\sin^2(\phi) - \cos^2(\phi) - 2i\cos(\theta)\sin(\phi)\cos(\phi)] + \sin^2(\theta) \big) \\
&= -\frac{(\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)) - (\sin^2(\phi) + \cos^2(\phi))}{2} = 0
\end{aligned}$$

$$
\begin{aligned}
\varepsilon_1^* \cdot \varepsilon_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \cos\theta\cos\phi + i\sin\phi & \cos\theta\sin\phi - i\cos\phi & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
&= -\frac{1}{\sqrt{2}} \big([\cos(\theta)\sin(\theta)\cos^2(\phi) + i\sin(\theta)\sin(\phi)\cos(\phi)] \\
&\qquad + [\sin(\theta)\cos(\theta)\sin^2(\phi) - i\sin(\theta)\sin(\phi)\cos(\phi)] - \sin(\theta)\cos(\theta) \big) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_2^* \cdot \varepsilon_2 &= \frac{1}{2} \begin{pmatrix} 0 & \cos\theta\cos\phi - i\sin\phi & \cos\theta\sin\phi + i\cos\phi & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \cos\theta\cos\phi + i\sin\phi \\ \cos\theta\sin\phi - i\cos\phi \\ -\sin\theta \end{pmatrix} \\
&= -\frac{1}{2} \left([\cos^2(\theta)\cos^2(\phi) + \sin^2(\phi)] + [\cos^2(\theta)\sin^2(\phi) + \cos^2(\phi)] + \sin^2(\theta) \right) \\
&= -\frac{(\cos^2(\theta)\cos^2(\phi) + \cos^2(\theta)\sin^2(\phi) + \sin^2(\theta)) + (\sin^2(\phi) + \cos^2(\phi))}{2} = -1
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_2^* \cdot \varepsilon_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & \cos\theta\cos\phi - i\sin\phi & \cos\theta\sin\phi + i\cos\phi & -\sin\theta \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
&= -\frac{1}{\sqrt{2}} \big([\cos(\theta)\sin(\theta)\cos^2(\phi) - i\sin(\theta)\sin(\phi)\cos(\phi)] \\
&\qquad + [\sin(\theta)\cos(\theta)\sin^2(\phi) + i\sin(\theta)\sin(\phi)\cos(\phi)] - \sin(\theta)\cos(\theta) \big) \\
&= 0
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_3^* \cdot \varepsilon_3 &= \begin{pmatrix} 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{pmatrix} \\
&= -\sin^2(\theta)\cos^2(\phi) - \sin^2(\theta)\sin^2(\phi) - \cos^2(\theta) = -1
\end{aligned}
$$

completando la comprobación.